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# Controllability of nonlocal non-autonomous neutral differential systems including non-instantaneous impulsive effects in $\mathbb{R}^n$

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## Abstract

This manuscript involves a class of first-order controllability results for nonlocal non-autonomous neutral differential systems with non-instantaneous impulses in the space  $\mathbb{R}^n$ . Sufficient conditions guaranteeing the controllability of mild solutions are set up. Concept of evolution family and Rothe's fixed point theorem are employed to achieve the required results. A model is investigated to delineate the adequacy of the results.

## 1 Introduction

Some real-life procedures can be described by quick changes in their state. A portion of these progressions start incautiously and are not insignificant. For changes, for example, these, scientific models called non-instantaneous differential conditions are made. These models offer ascent to another, half and half dynamical system that can be utilized for various purposes. The investigation on non-abrupt impulses be one of present concern of several researchers [1, 2, 8, 13, 4, 6, 9, 16, 14]. Hernandez and O'Regan [8] and Pierri et al. [13], initially analyzed Cauchy problems for first order evolution equations with non-instantaneous impulses. Lately, Wang and Feckan [17] generalized

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the results of [8] under different conditions. For further information, please reader to refer [8, 13, 6, 9].

The nonlocal issue was driven by physical issues. Moreover, it has been shown that nonlocal problems have a stronger impact on applications than the classic Cauchy problems. It is used, for instance, to depict mathematical models for the evolution of multiple phenomena such as nonlocal neutral networks, nonlocal pharmacokinetics, and non-local pollution (see [5, 7]).

The idea of controllability indicates the capacity to move a framework around in its whole configuration space utilizing just certain allowable controls. The perfect mathematical definition shifts marginally inside the structure or the sort of models applied. Nowadays, there are numerous intriguing uses of control concept, for instance, reactor, biological networks, chemical technology and power system control, in general, concept of control theory used in different branches of science and technology etc [15, 3, 12, 10, 11]. In [10], author discussed the controllability of non-autonomous systems with the help of Rothe's fixed point theorem. Further, the same author [11] studied the controllability of semilinear impulsive differential systems in  $\mathbb{R}^n$  under some suitable conditions and Rothe's fixed point theorem. Lately, Malik et al. [12] analyzed the controllability results for non-autonomous nonlinear differential systems with non-instantaneous impulses under suitable fixed point theorem.

As per the authors knowledge, there is no work yet reported on non-instantaneous impulsive neutral non-autonomous differential system with non-local conditions in  $\mathbb{R}^n$ . For this reason, to bridge this crevice, we deal with a nonlocal non-autonomous neutral differential systems including non-abrupt impulsive effects in  $\mathbb{R}^n$ .

Inspired by the above mentioned works [17, 12, 11], in this manuscript, we discuss a non-autonomous nonlocal neutral differential system along non-abrupt impulsive effects of the form

$$\frac{d}{dt}[w(t) - \mathcal{H}(t, w(t))] = A(t)[w(t) - \mathcal{H}(t, w(t))] + B(t)u(t) + \mathcal{F}(t, w(t)),$$

$$t \in \bigcup_{q=0}^p (s_q, t_{q+1}), \tag{1.1}$$

$$w(t) = \xi_q(t, w(t_q^-)), \quad t \in \bigcup_{q=1}^p (t_q, s_q], \tag{1.2}$$

$$w(0) = w_0 + r(w) \in \mathbb{R}^n, \tag{1.3}$$

where  $w(t) \in \mathbb{R}^n, 0 = s_0 = t_0 < t_1 < s_1 < t_2, \dots, t_p < s_p < t_{p+1} = T < \infty, u(t) \in \mathbb{R}^m$ . From equation (1.2), we notice that  $w \in C((t_q, t_{q+1}], \mathbb{R}^n), q = 0, 1, \dots, p$  and we can find  $w(t_q^-)$  and  $w(t_q^+), q = 1, 2, \dots, p$  with  $w(t_q^-) =$

$w(t_q)$ . The functions  $\xi_q(t, w(t_q^-))$  constitute the non-instantaneous impulses during the interval  $(t_q, s_q]$ ,  $q = 1, 2, \dots, p$ , thus impulses at  $t_q$  have some duration, namely on intervals  $(t_q, s_q]$ . Moreover  $A(t)$  and  $B(t)$  defined as a matrices of  $n \times n$  and  $n \times m$ , separately. The functions specified in (1.1)-(1.3) are defined further section appropriately.

The scheme about manuscript is as stated. Within the to begin with and second segments, we present the presentation, crucial definitions and ground-work which are utilized all through this manuscript. In section 3, authors demonstrate the controllability of the linear control framework comparing the issue (1.1)-(1.3) without impulses. Further, we built up the adequate setting for the controllability of nonlocal non-autonomous neutral differential systems with non-instantaneous impulses of the system (1.1)-(1.3) with the help of Rothe's fixed point theorem in section 4. In the last section, an example is given to show the application of the obtained results.

## 2 Preliminary Results and Hypotheses

In this section, we recall some basic definitions, notations and theorem which will be used throughout this manuscript.

Let  $\mathcal{PC}([0, T], \mathbb{R}^n)$  be the space of piecewise continuous functions. That is,  $\mathcal{PC}([0, T], \mathbb{R}^n) = \{w : \mathcal{J} = [0, T] \rightarrow \mathbb{R}^n : w \in C((t_j, t_{j+1}], \mathbb{R}^n), j = 0, 1, 2, \dots, p, \text{ and we can find } w(t_j^-) \text{ and } w(t_j^+), j = 1, 2, \dots, p \text{ with } w(t_j^-) = w(t_j)\}$ , supplied with the supremum criterion

$$\|w\|_0 = \sup_{t \in \mathcal{J}} \|w(t)\|_{\mathbb{R}^n}.$$

Presently, we treat  $\mathcal{PC}(\mathcal{J}) \times \mathcal{C}(\mathcal{J}) = \mathcal{PC}(\mathcal{J}, \mathbb{R}^n) \times \mathcal{C}(\mathcal{J}, \mathbb{R}^m)$  is a Banach space supplied with the subsequent criterion

$$\|(w, u)\|_0 = \|x\|_0 + \|u\|_0.$$

In  $\mathbb{R}^n \times \mathbb{R}^m$ , we treat the subsequent criterion

$$\|(w, u)\|_1 = \|w\|_{\mathbb{R}^n} + \|u\|_{\mathbb{R}^m} = \|w\| + \|u\|, \quad \forall (w, u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Each  $w \in \mathcal{PC}(\mathcal{J})$ , we characterize the subsequent criterion

$$\|\mathcal{H}(\cdot, w)\|_0 = \sup_{t \in \mathcal{J}} \|\mathcal{H}(t, w(t))\|_{\mathbb{R}^n}$$

and

$$\|\mathcal{F}(\cdot, w)\|_0 = \sup_{t \in \mathcal{J}} \|\mathcal{F}(t, w(t))\|_{\mathbb{R}^n}.$$

**Definition 2.1.** *The problem (1.1)-(1.3) is said to be controllable on  $\mathcal{J}$  if for every  $w(0), w_1 \in \mathbb{R}^n$ , we can find  $u \in \mathcal{L}^2(\mathcal{J}, \mathbb{R}^n)$  in a way that the solution  $w(t)$  of the given problem (1.1)-(1.3) analogous to  $u$  fulfills*

$$w(0) = w_0 + r(w) \quad \text{and} \quad w(T) = w_1.$$

To establish the primary result of the problem (1.1)-(1.3), now, we list the subsequent assumptions:

(H1) The function  $\mathcal{F} : \mathcal{J}_1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\mathcal{J}_1 = \cup_{q=0}^p [s_q, t_{q+1}]$  is continuous, in a way that

$$\|\mathcal{F}(t, w)\|_{\mathbb{R}^n} \leq \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0, \quad t \in \mathcal{J}_1, \quad w \in \mathbb{R}^n,$$

with  $0 < \beta_0 \leq 1$ .

(H2) The function  $\mathcal{H} : \mathcal{J}_1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and we can find a positive constant  $\mathcal{L}_{\mathcal{H}}$  in a way that

$$\|\mathcal{H}(t, w_1) - \mathcal{H}(t, w_2)\|_{\mathbb{R}^n} \leq \mathcal{L}_{\mathcal{H}} \|w_1 - w_2\|_{\mathbb{R}^n},$$

for every  $w_1, w_2 \in \mathbb{R}^n$ . Also, we can find a positive constant  $C_{\mathcal{H}}$  such that  $\|\mathcal{H}(t, w)\| \leq C_{\mathcal{H}}$ , for all  $t \in \mathcal{J}_1$  and  $w \in \mathbb{R}^n$ .

(H3) The function  $r : \mathcal{C}(\mathcal{J}, \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is continuous in a way that

$$\|r(w)\|_{\mathbb{R}^n} \leq c \|w\|_{\mathcal{C}}^{\eta} + c_0,$$

with  $0 < \eta \leq 1$ .

(H4) The function  $r : \mathcal{C}(\mathcal{J}, \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is continuous and we can find a positive constant  $c_1$  in a way that

$$\|r(w_1) - r(w_2)\|_{\mathbb{R}^n} \leq c_1 \|w_1 - w_2\|_{\mathcal{C}}.$$

(H5) The functions  $\xi_q : I_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $I_q = [t_q, s_q]$ ,  $q = 1, 2, \dots, p$  are continuous such that

$$\|\xi_q(t, x)\|_{\mathbb{R}^n} \leq \gamma_q \|x\|_{\mathbb{R}^n}^{\beta_q} + \delta_q, \quad t \in I_q, \quad x \in \mathbb{R}^n,$$

with  $0 < \beta_q \leq 1$ ,  $q = 1, 2, \dots, p$ .

(H6) The functions  $\xi_q : I_q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous and there exist constants  $\mathcal{L}_{\xi_q} > 0$ ,  $q = 1, 2, \dots, p$ , in ways that

$$\|\xi_q(t, w_1) - \xi_q(s, w_2)\|_{\mathbb{R}^n} \leq \mathcal{L}_{\xi_q} (|t - s| + \|w_1 - w_2\|_{\mathbb{R}^n}), \quad \forall t, s \in I_q, \quad w_1, w_2 \in \mathbb{R}^n.$$

Now, we are in a position to define the mild solution for the system (1.1)-(1.3).

**Definition 2.2.** A function  $w(t) \in \mathbb{R}^n$  is called a mild solution of the system (1.1)-(1.3) if  $w(0) = w_0 + r(w)$ ,  $w(t) = \xi_q(t, w(t_q^-))$  for all  $t \in (t_q, s_q]$ ,  $q = 1, 2, \dots, p$  and  $w(t)$  is the solution of the subsequent integral equations

$$w(t) = U(t, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] + \mathcal{H}(t, w(t)) + \int_0^t U(t, s)B(s)u(s)ds + \int_0^t U(t, s)\mathcal{F}(s, w(s))ds, \quad \forall t \in [0, t_1], \quad (2.1)$$

$$w(t) = U(t, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t, w(t_q^-)) + \int_{s_q}^t U(t, s)B(s)u(s)ds + \int_{s_q}^t U(t, s)\mathcal{F}(s, w(s))ds, \quad \forall t \in (s_q, t_{q+1}], \quad q = 1, 2, \dots, p. \quad (2.2)$$

### 3 Controllability for Linear System

Before we establish the controllability result of the problem (1.1)-(1.3), first, we discuss the subsequent problem having no impulsive conditions.

We accept that the nonlocal non-autonomous linear problem

$$w'(t) = A(t)w(t) + B(t)u(t), \quad t \in \mathcal{J}, \quad (3.1)$$

$$w(0) = w_0 + r(w) \in \mathbb{R}^n, \quad (3.2)$$

admits unique solution given by

$$w(t) = U(t, 0)[w_0 + r(w)] + \int_0^t U(t, s)B(s)u(s)ds, \quad \forall t \in \mathcal{J}, \quad (3.3)$$

where  $U(t, s) = \Psi(t)\Psi^{-1}(s)$  and  $\Psi(t)$  is the fundamental matrix of the system

$$w'(t) = A(t)w(t).$$

The matrix  $\Psi(t)$  fulfills

$$\Psi'(t) = A(t)\Psi(t)$$

$$\Psi(0) = I_{\mathbb{R}^n},$$

where  $I_{\mathbb{R}^n}$  is the  $n \times n$  identity matrix. Consequently there exist constants  $\mathcal{M} > 0$  and  $\omega > 0$  in a way that

$$\|U(t, s)\| \leq \mathcal{M}e^{\omega(t-s)}, \quad 0 \leq s \leq t \leq T.$$

Presently, we designate the map  $\Upsilon : \mathcal{L}^2(\mathcal{J}, \mathbb{R}^m) \rightarrow \mathbb{R}^n$  is denoted by

$$\Upsilon u = \int_0^T U(T, s)B(s)u(s)ds.$$

The adjoint operator  $\Upsilon^* : \mathbb{R}^n \rightarrow \mathcal{L}^2(\mathcal{J}, \mathbb{R}^m)$  of the operator  $\Upsilon$  is denoted by

$$(\Upsilon^*w)(s) = B^*(s)U^*(T, s)w.$$

In this situation, the controllability Grammian matrix  $G_0^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is given by

$$G_0^T w = \int_0^T U(T, s)B(s)B^*(s)U^*(T, s)w(s)ds. \quad (3.4)$$

Based on the above discussion, the problem (3.1)-(3.2) is controllable if and only if there exists a  $\delta > 0$  in a way that

$$\langle G_0^T w, w \rangle \geq \delta \|w\|^2, \quad \forall w \in \mathbb{R}^n, \quad (3.5)$$

then  $\|(G_0^T)^{-1}\| \leq \frac{1}{\delta}$ .

Therefore, the operator  $\bar{\Upsilon} : \mathbb{R}^n \rightarrow \mathcal{L}^2(\mathcal{J}, \mathbb{R}^m)$  defined by

$$\bar{\Upsilon}w = B^*(\cdot)U^*(T, \cdot)(G_0^T)^{-1}w = \Upsilon^*(\Upsilon\Upsilon^*)^{-1}w \quad (3.6)$$

is called the steering operator and is an inverse for the right of  $\Upsilon$ , include that  $\Upsilon\bar{\Upsilon} = I$ .

Presently, we characterize the steering control that move  $w(0)$  to  $w_1$  is given by

$$u(t) = B^*(t)U^*(T, t)(G_0^T)^{-1}[w_1 - U(T, 0)(w_0 + r(w))].$$

For additional concept on this theory, we suggest the reader to refer [10, 11, 12].

## 4 Main Results

In this section, we present and prove the controllability results for the system (1.1)-(1.3) along with Rothe's fixed point theorem. Before, we prove the main theorem of this manuscript, initially, we need to prove the following two auxiliary results.

**Lemma 4.1.** *If the conditions (H1)–(H3) are hold, then the control function for the system (1.1)-(1.3) has  $\|u(t)\| \leq Q_1, \forall t \in [0, t_1]$ , where*

$$Q_1 = \frac{\widetilde{\mathcal{M}}\|B\|}{\delta} \left[ \|w_{t_1}\| + \mathcal{M}_1\{\|w_0\| + c\|w\|_{\mathcal{C}}^\eta + c_0 - \mathcal{M}_2\} + \mathcal{M}_2 \right. \\ \left. + \mathcal{M}_3\{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} \right],$$

where  $\widetilde{\mathcal{M}} = \mathcal{M}e^{\omega(t_1-t)}, \mathcal{M}_1 = \mathcal{M}e^{\omega t_1}, \mathcal{M}_2 = C_{\mathcal{H}}$  and  $\mathcal{M}_3 = \frac{\mathcal{M}}{\omega}[e^{\omega t_1} - 1]$ .

*Proof.* For all  $t \in [0, t_1]$ , presently, we characterize the control function  $u(t)$ , we sustain

$$u(t) = B^*(t)U^*(t_1, t)(G_{s_0}^{t_1})^{-1}\widetilde{P}(w), \quad (4.1)$$

where

$$\widetilde{P}(w) = w_{t_1} - U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] - \mathcal{H}(t_1, w(t_1)) \\ - \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds.$$

By the mild solution (2.1), we have final state at  $t = t_1$

$$w(t_1) = U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] + \mathcal{H}(t_1, w(t_1)) \\ + \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds + \int_0^{t_1} U(t_1, s)B(s)u(s)ds \\ = U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] + \mathcal{H}(t_1, w(t_1)) \\ + \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds + (G_{s_0}^{t_1})(G_{s_0}^{t_1})^{-1} \left[ w_{t_1} - U(t_1, 0)[w_0 + r(w) \right. \\ \left. - \mathcal{H}(0, w(0))] - \mathcal{H}(t_1, w(t_1)) - \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds \right] \\ = w_{t_1}.$$

Thus, the control (4.1) is appropriate for  $t \in [0, t_1]$ .

Presently, we sustain:

$$\begin{aligned} \|u(t)\| &\leq \frac{\|B\|\mathcal{M}e^{\omega(t_1-t)}}{\delta} \left[ \|w_{t_1}\| + \mathcal{M}e^{\omega t_1} \{ \|w_0\| + c\|w\|_{\mathcal{E}}^{\eta} + c_0 - C_{\mathcal{H}} \} + C_{\mathcal{H}} \right. \\ &\quad \left. + \int_0^{t_1} \mathcal{M}e^{\omega(t_1-s)} \{ \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0 \} ds \right] \\ &\leq \frac{\overline{\mathcal{M}}\|B\|}{\delta} \left[ \|w_{t_1}\| + \mathcal{M}_1 \{ \|w_0\| + c\|w\|_{\mathcal{E}}^{\eta} + c_0 - \mathcal{M}_2 \} + \mathcal{M}_2 \right. \\ &\quad \left. + \mathcal{M}_3 \{ \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0 \} \right]. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 4.2.** *If the conditions (H1) – (H3) and (H5) are hold, then the control function for the system (1.1)-(1.3) has  $\|u(t)\| \leq Q_q$ ,  $\forall t \in [s_q, t_{q+1}]$ ,  $q = 1, 2, \dots, p$ , where*

$$\begin{aligned} Q_q &= \frac{\overline{\mathcal{M}}\|B\|}{\delta_q} \left[ \|w_{t_{q+1}}\| + \mathcal{M}_4 \{ \gamma_q \|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q \} + (\mathcal{M}_4 + 1) \mathcal{M}_2 \right. \\ &\quad \left. + \mathcal{M}_5 \{ \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0 \} \right], \end{aligned}$$

where  $\overline{\mathcal{M}} = \mathcal{M}e^{\omega(t_{q+1}-t)}$ ,  $\mathcal{M}_4 = \mathcal{M}e^{\omega(t_{q+1}-s_q)}$  and  $\mathcal{M}_5 = \frac{\mathcal{M}}{\omega} [e^{\omega(t_{q+1}-s_q)} - 1]$ ,  $q = 1, 2, \dots, p$ .

*Proof.* For all  $t \in [s_q, t_{q+1}]$ ,  $q = 1, 2, \dots, p$ , presently, we characterize the control function  $u(t)$ , we sustain

$$u(t) = B^*(t)U^*(t_{q+1}, t)(G_{s_q}^{t_{q+1}})^{-1}\tilde{P}_1(w), \quad (4.2)$$

where

$$\begin{aligned} \tilde{P}_1(w) &= w_{t_{q+1}} - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \\ &\quad - \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds, \end{aligned}$$

and

$$G_{s_q}^{t_{q+1}}w = \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)B(s)B^*(s)U^*(t_{q+1}, s)w(s)ds \text{ and } \xi_0(0, \cdot) = w_0 + r(w).$$



We replacement the estimation of the  $u(t)$  from (4.2) in (2.2), and supplant  $t$  by  $t_{q+1}$ ,  $q = 1, 2, \dots, p$ ,  $w(t_{p+1}) = w_{t_{p+1}} = w_1$ , we obtain

$$\begin{aligned}
 w(t_{q+1}) &= U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t_{q+1}, w(t_q^-)) \\
 &\quad + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)B(s)u(s)ds \\
 &= U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t_{q+1}, w(t_q^-)) \\
 &\quad + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)B(s)B^*(s) \\
 &\quad (\times)U^*(t_{q+1}, s)(G_{s_q}^{t_{q+1}})^{-1} \left[ w_{t_{q+1}} - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) \right. \\
 &\quad \left. - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \right. \\
 &\quad \left. - \int_{s_q}^{t_{q+1}} U(t_{q+1}, \eta)\mathcal{F}(\eta, w(\eta))d\eta \right] ds \\
 &= U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t_{q+1}, w(t_q^-)) \\
 &\quad + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds + (G_{s_q}^{t_{q+1}})(G_{s_q}^{t_{q+1}})^{-1} \left[ w_{t_{q+1}} \right. \\
 &\quad \left. - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \right. \\
 &\quad \left. - \int_{s_q}^{t_{q+1}} U(t_{q+1}, \eta)\mathcal{F}(s, w(s))ds \right] \\
 &= w_{t_{q+1}}.
 \end{aligned}$$

Hence, the control function  $u(t)$  given by (4.2) drive the state function  $w(t)$  from starting state  $w(0)$  to definite state  $w_1$ . Presently, we obtain

$$\begin{aligned}
 &\|u(t)\| \\
 &\leq \frac{\|B\|\mathcal{M}e^{\omega(t_{q+1}-t)}}{\delta_q} \left[ \|w_{t_{q+1}}\| + \|U(t_{q+1}, s_q)\| \left[ \|\xi_q(s_q, w(t_q^-))\| + \|\mathcal{H}(s_q, w(t_q^-))\| \right] \right. \\
 &\quad \left. + \|\mathcal{H}(t_{q+1}, w(t_q^-))\| + \int_{s_q}^{t_{q+1}} \|U(t_{q+1}, s)\|\mathcal{F}(s, w(s))\|ds \right] \\
 &\leq \frac{\overline{\mathcal{M}}\|B\|}{\delta_q} \left[ \|w_{t_{q+1}}\| + \mathcal{M}_4\{\gamma_q\|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\} + (\mathcal{M}_4 + 1)\mathcal{M}_2 + \mathcal{M}_5\{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} \right] \\
 &\leq Q_q, \quad q = 1, 2, \dots, p.
 \end{aligned}$$

The proof is now completed.  $\square$

Now, we define the operator  $\tilde{\Upsilon} : \mathcal{PC}(\mathcal{J}, \mathbb{R}^n) \times (\mathcal{J}, \mathbb{R}^m) \rightarrow \mathcal{PC}(\mathcal{J}, \mathbb{R}^n) \times$

$(\mathcal{J}, \mathbb{R}^m)$  by the subsequent form

$$(z, v) = (\Upsilon_1(w, u), \Upsilon_2(w, u)) = \tilde{\Upsilon}(w, u),$$

where

$$\begin{aligned} z(t) = \Upsilon_1(w, u)(t) &= U(t, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] + \mathcal{H}(t, w(t)) \\ &+ \int_0^t U(t, s)\mathcal{F}(s, w(s))ds + \int_0^t U(t, s)B(s)[\tilde{\Upsilon}\lambda_0(w, u)]ds, \quad \forall t \in [0, t_1] \end{aligned} \quad (4.3)$$

$$z(t) = \Upsilon_1(w, u)(t) = \xi_q(t, w(t_q^-)), \quad \forall t \in (t_q, s_q], \quad q = 1, 2, \dots, p, \quad (4.4)$$

$$\begin{aligned} z(t) = \Upsilon_1(w, u)(t) &= U(t, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t, w(t_q^-)) \\ &+ \int_{s_q}^t U(t, s)\mathcal{F}(s, w(s))ds + \int_{s_q}^t U(t, s)B(s)[\tilde{\Upsilon}\lambda_q(w, u)]ds, \\ &\forall t \in (s_q, t_{q+1}], \quad q = 1, 2, \dots, p, \end{aligned} \quad (4.5)$$

$$v(t) = \Upsilon_2(w, u)(t) = [\tilde{\Upsilon}\lambda_0(w, u)] = B^*(t)U^*(t_1, t)(G_{s_0}^{t_1})^{-1}[\lambda_0(w, u)], \quad \forall t \in [0, t_1], \quad (4.6)$$

$$\begin{aligned} v(t) = \Upsilon_2(w, u)(t) &= [\tilde{\Upsilon}\lambda_q(w, u)] = B^*(t)U^*(t_{q+1}, t)(G_{s_q}^{t_{q+1}})^{-1}[\lambda_q(w, u)], \\ &\forall t \in [s_q, t_{q+1}], \quad q = 1, 2, \dots, p, \end{aligned} \quad (4.7)$$

with

$$\begin{aligned} \lambda_0(w, u) &= w_{t_1} - U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] - \mathcal{H}(t_1, w(t_1)) \\ &- \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \lambda_q(w, u) &= w_{t_{q+1}} - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \\ &- \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds, \quad q = 1, 2, \dots, p. \end{aligned} \quad (4.9)$$

**Theorem 4.1.** *Suppose the conditions (H1) – (H6) are fulfilled. The system (1.1)-(1.3) is controllable if and only if for all  $w(0), w_1 \in \mathbb{R}^n$ , then operator  $\tilde{\Upsilon}(w, u)$  estimated in (4.3)-(4.9) has a fixed point. i.e.,*

$$\tilde{\Upsilon}(w, u) = (w, u).$$

Now, we are in a position to present and prove a main theorem of this manuscript.

**Theorem 4.2.** *Assume that the hypotheses (H1) – (H6) are hold and the analogous linear system is controllable on  $\mathcal{J}$ . If  $0 < \beta_q < 1$ ,  $q = 0, 1, 2, \dots, p$  and  $0 < \eta < 1$ , then the problem (1.1)-(1.3) is controllable on  $\mathcal{J}$ . Furthermore, we can find a control  $u \in (\mathcal{J}, \mathbb{R}^m)$  such that for a given  $w(0), w_1 \in \mathbb{R}^n$  the corresponding solution  $w(\cdot)$  of system (1.1)-(1.3) fulfills*

$$\begin{aligned} w_{t_1} &= U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] + \mathcal{H}(t_1, w(t_1)) \\ &\quad + \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds + \int_0^{t_1} U(t, s)B(s)u(s)ds, \\ w_{t_{q+1}} &= U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] + \mathcal{H}(t_{q+1}, w(t_q^-)) \\ &\quad + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds + \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)B(s)u(s)ds, \\ &\quad q = 1, 2, \dots, p, \end{aligned}$$

and

$$\begin{aligned} u(t) &= B^*(t)U^*(t_1, t)(G_{s_0}^{t_1})^{-1}[\lambda_0(w, u)], \quad \forall t \in [0, t_1]; \\ u(t) &= B^*(t)U^*(t_{q+1}, t)(G_{s_q}^{t_{q+1}})^{-1}[\lambda_q(w, u)], \quad \forall t \in [s_q, t_{q+1}], \quad q = 1, 2, \dots, p, \end{aligned}$$

where  $\lambda_0(w, u)$  and  $\lambda_q(w, u)$  are same as defined in (4.8) and (4.9) respectively.

*Proof.* For better readability, we break the argument of this theorem within 4 steps.

**Step 1:** The operator  $\tilde{\Upsilon}$  is continuous.

The continuity of  $\Upsilon_1$  and  $\Upsilon_2$  are directly pursue against the continuity of the functions [(H1)-(H6)]  $\mathcal{H}(t, w)$ ,  $\mathcal{F}(t, w)$ ,  $r(w)$ ,  $\xi_q(t, w)$ ,  $q = 1, 2, \dots, p$ , and  $\lambda_q$ ,  $q = 1, 2, \dots, p$  respectively (since  $\tilde{\Upsilon}$  is well defined earlier).

**Step 2:** Now, we intend to demonstrate that the operator  $\tilde{\Upsilon}$  is compact. Allow  $D$  be a bounded subset of  $\mathcal{PC}(\mathcal{J}, \mathbb{R}^n) \times \mathcal{C}(\mathcal{J}, \mathbb{R}^m)$ . It pursue that

$$\begin{aligned} \|\mathcal{F}(\cdot, w)\|_0 &\leq R_1, \quad \|\mathcal{G}(\cdot, w)\|_0 \leq R_2, \quad \|r(w)\| \leq R_3, \\ \|(G_{s_q}^{t_{q+1}})^{-1}\lambda_q(w, u)\| &\leq R_{q+4}, \quad q = 0, 1, \dots, p, \\ \|\xi_q(\cdot, w)\|_0 &\leq R_{p+q+5}, \quad q = 1, 2, \dots, p, \quad \forall (w, u) \in D. \end{aligned}$$

Therefore,  $\tilde{\Upsilon}$  is bounded.

Next, we prove that the operator  $\tilde{\Upsilon}$  is equi-continuous on  $\mathcal{J}$ . To prove this, let  $0 \leq \tau_2 < \tau_3 \leq t_1$ , we sustain

$$\begin{aligned} &\|\tilde{\Upsilon}(w, u)(\tau_3) - \tilde{\Upsilon}(w, u)(\tau_2)\|_1 \\ &= \|\Upsilon_1(w, u)(\tau_3) - \Upsilon_1(w, u)(\tau_2)\| + \|\Upsilon_2(w, u)(\tau_3) - \Upsilon_2(w, u)(\tau_2)\|. \end{aligned}$$

Now,

$$\begin{aligned}
 & \|\Upsilon_1(w, u)(\tau_3) - \Upsilon_1(w, u)(\tau_2)\| \\
 & \leq \|U(\tau_3, 0) - U(\tau_2, 0)\| [\|w_0\| + \|r(w)\| + \mathcal{H}(0, w(0))] \\
 & \quad + \|\mathcal{H}(\tau_3, w(\tau_3)) - \mathcal{H}(\tau_2, w(\tau_2))\| + \int_{\tau_2}^{\tau_3} \|U(\tau_3, s)\| \|B(s)\| \|u(s)\| ds \\
 & \quad + \int_0^{\tau_2} \|U(\tau_3, s) - U(\tau_2, s)\| \|B(s)\| \|u(s)\| ds \\
 & \quad + \int_{\tau_2}^{\tau_3} \|U(\tau_3, s)\| \|\mathcal{F}(s, w(s))\| ds \\
 & \quad + \int_0^{\tau_2} \|U(\tau_3, s) - U(\tau_2, s)\| \|\mathcal{F}(s, w(s))\| ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \|\Upsilon_2(w, u)(\tau_3) - \Upsilon_2(w, u)(\tau_2)\| \\
 & \leq \|B^*(\tau_3)U^*(t_1, \tau_3) - B^*(\tau_1)U^*(t_1, \tau_2)\| [\|(G_{s_0}^{t_1})^{-1}\| \|\lambda_0(w, u)\|].
 \end{aligned}$$

Let  $s_q < \tau_2 < \tau_3 \leq t_{q+1}$ ,  $q = 1, 2, \dots, p$ , we have

$$\begin{aligned}
 & \|\tilde{\Upsilon}(w, u)(\tau_3) - \tilde{\Upsilon}(w, u)(\tau_2)\| \\
 & \leq \|\Upsilon_1(w, u)(\tau_3) - \Upsilon_1(w, u)(\tau_2)\| + \|\Upsilon_2(w, u)(\tau_3) - \Upsilon_2(w, u)(\tau_2)\|.
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \|\Upsilon_1(w, u)(\tau_3) - \Upsilon_1(w, u)(\tau_2)\| \\
 & \leq \|U(\tau_3, s_q) - U(\tau_2, s_q)\| [\|\xi_q(s_q, w(t_q^-))\| + \|\mathcal{H}(s_q, w(t_q^-))\|] \\
 & \quad + \|\mathcal{H}(\tau_3, w(t_q^-)) - \mathcal{H}(\tau_2, w(t_q^-))\| \\
 & \quad + \int_{s_q}^{\tau_2} \|U(\tau_3, s) - U(\tau_2, s)\| \|B(s)\| \|u(s)\| ds + \int_{\tau_2}^{\tau_3} \|U(\tau_3, s)\| \|B(s)\| \|u(s)\| ds \\
 & \quad + \int_{s_q}^{\tau_2} \|U(\tau_3, s) - U(\tau_2, s)\| \|\mathcal{F}(s, w(s))\| ds + \int_{\tau_2}^{\tau_3} \|U(\tau_3, s)\| \|\mathcal{F}(s, w(s))\| ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \|\Upsilon_1(w, u)(\tau_3) - \Upsilon_1(w, u)(\tau_2)\| \\
 & \leq \|B^*(\tau_3)U^*(t_{q+1}, \tau_3) - B^*(\tau_1)U^*(t_{q+1}, \tau_2)\| [\|(G_{s_q}^{t_{q+1}})^{-1}\| \|\lambda_i(w, u)\|].
 \end{aligned}$$

Let  $t_q < \tau_2 < \tau_3 \leq s_q$ ,  $q = 1, 2, \dots, p$ , we have

$$\|\tilde{\Upsilon}(w, u)(\tau_3) - \tilde{\Upsilon}(w, u)(\tau_2)\| \leq \|\xi_q(\tau_3, w(t_q^-)) - \xi_q(\tau_2, w(t_q^-))\| \leq \mathcal{L}_{\xi_q} |\tau_3 - \tau_2|.$$

Presume, if we select a sequence  $\{\varphi_j\}$  against  $\tilde{\Upsilon}(D)$  which is equi-continuous and uniformly bounded on the interval  $[0, t_1]$ . Subsequently, by utilizing the Arzela theorem, there is a sub-sequence  $\{\varphi_j^1\}$  of  $\{\varphi_j\}$  which is uniformly convergent on  $[0, t_1]$ .

Presently, we consider sequence  $\{\varphi_j^1\}$  on the interval  $(t_1, t_2]$ . Since the sequence  $\{\varphi_j^1\}$  is equi-continuous and uniformly bounded on  $(t_1, t_2]$ . Thus, by making use of Arzela theorem, there is a sub-sequence  $\{\varphi_j^2\}$  which is uniformly convergent on  $[0, t_2]$ .

Continuing this process for the intervals  $(t_2, t_3], (t_3, t_4], \dots, (t_p, t_{p+1}]$ , we observe that the sequence  $\{\varphi_j^{p+1}\}$  converges uniformly on the interval  $\mathcal{J}$ . From the above discussion, we confirm that the operator  $\tilde{\Upsilon}$  is compact.

**Step 3:** In this case, we intend to demonstrate that the subsequent inequality

$$\lim_{\|(w,u)\| \rightarrow \infty} \frac{\|\tilde{\Upsilon}(w,u)\|}{\|(w,u)\|} = 0,$$

where  $\|(w,u)\|$  is the norm in the space  $\mathcal{P}\mathcal{C}(\mathcal{J}, \mathbb{R}^n) \times \mathcal{C}(\mathcal{J}, \mathbb{R}^m)$ .

To prove this, we need to furnish the following estimations:

$$\begin{aligned} \|\lambda_0(w,u)\| &\leq \|w_{t_1}\| + \|U(t_1,0)\|(\|w_0\| + \|r(w)\| + \|\mathcal{H}(0,w(0))\|) \\ &\quad + \|\mathcal{H}(t_1,w(t_1))\| + \int_0^{t_1} \|U(t_1,s)\| \|\mathcal{F}(s,w(s))\| ds \\ &\leq \|w_{t_1}\| + \mathcal{M}_1\|w_0\| + \mathcal{M}_2(\mathcal{M}_1 + 1) + \mathcal{M}_1\{c\|w\|_{\mathcal{C}}^\eta + c_0\} \\ &\quad + \mathcal{M}_3\{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\}, \end{aligned}$$

where  $\mathcal{M}_1$ ,  $\mathcal{M}_2$  and  $\mathcal{M}_3$  are same as defined in Lemma 4.1.

$$\begin{aligned} \|\lambda_q(w,u)\| &\leq \|w_{t_{q+1}}\| + \|U(t_{q+1},s_q)\|(\|\xi_q(s_q,w(t_q^-))\| + \|\mathcal{H}(s_q,w(t_q^-))\|) \\ &\quad + \|\mathcal{H}(t_{q+1},w(t_q^-))\| + \int_{s_q}^{t_{q+1}} \|U(t_{q+1},s)\| \|\mathcal{F}(s,w(s))\| ds \\ &\leq \|w_{t_{q+1}}\| + \mathcal{M}_2(\mathcal{M}_4 + 1) + \mathcal{M}_4\{\gamma_q\|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\} \\ &\quad + \mathcal{M}_5\{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\}, \quad q = 1, 2, \dots, p, \end{aligned}$$

where  $\mathcal{M}_4$  and  $\mathcal{M}_5$  are same as defined in Lemma 4.2.

$$\begin{aligned} &\|\Upsilon_2(w,u)\| \\ &\leq \tilde{\mathcal{M}}\|B\| \|(G_{s_0}^{t_1})^{-1}\| \{\|w_{t_1}\| + \mathcal{M}_1\|w_0\|\} + \tilde{\mathcal{M}}\|B\| \|(G_{s_0}^{t_1})^{-1}\| \{\mathcal{M}_2(1 + \mathcal{M}_1)\} \\ &\quad + \tilde{\mathcal{M}}\mathcal{M}_1\|B\| \|(G_{s_0}^{t_1})^{-1}\| \{c\|w\|_{\mathcal{C}}^\eta + c_0\} + \tilde{\mathcal{M}}\mathcal{M}_3\|B\| \{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\}, \quad t \in [0, t_1], \end{aligned}$$

where  $\widetilde{\mathcal{M}}$  is same as defined in Lemma 4.1.

$$\begin{aligned} & \|\Upsilon_2(w, u)\| \\ & \leq \overline{\mathcal{M}}\|B\| \|(G_{s_q}^{t_{q+1}})^{-1}\| \{\|w_{t_{q+1}}\| + (\mathcal{M}_4 + 1)\mathcal{M}_2\} + \overline{\mathcal{M}}\mathcal{M}_5\|B\| \|(G_{s_q}^{t_{q+1}})^{-1}\| \\ & \quad \cdot \{\gamma_q\|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\} + \overline{\mathcal{M}}\mathcal{M}_5\|B\| \|(G_{s_q}^{t_{q+1}})^{-1}\| \{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\}, \\ & \quad t \in (s_q, t_{q+1}], q = 1, 2, \dots, p, \end{aligned}$$

where  $\overline{\mathcal{M}}$  is same as defined in Lemma 4.2.

$$\begin{aligned} & \|\Upsilon_1(w, u)\| \\ & \leq [\widetilde{\mathcal{M}}_1 + \mathcal{M}_1\widetilde{\mathcal{M}}_3\|B\|^2 \|(G_{s_0}^{t_1})^{-1}\|] \{c\|w\|_{\mathcal{C}}^\eta + c_0\} + \widetilde{\mathcal{M}}_3[1 + \mathcal{M}_3\widetilde{\mathcal{M}}\|B\|^2 \|(G_{s_0}^{t_1})^{-1}\|] \\ & \quad (\times) \{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} + \Lambda_1, t \in [0, t_1], \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 = & \widetilde{\mathcal{M}}_1\|w_0\| + \mathcal{M}_2(\widetilde{\mathcal{M}}_1 + 1) + \widetilde{\mathcal{M}}_3\widetilde{\mathcal{M}}\|B\|^2 \|(G_{s_0}^{t_1})^{-1}\| [\|w_{t_1}\| + \mathcal{M}_1\|w_0\| \\ & + \{\mathcal{M}_2(1 + \mathcal{M}_1)\}]. \end{aligned}$$

$$\begin{aligned} \|\Upsilon_1(w, u)\| \leq & [\overline{\mathcal{M}}_1 + \overline{\mathcal{M}}_3\overline{\mathcal{M}}\mathcal{M}_4\|B\|^2 \|(G_{s_q}^{t_{q+1}})^{-1}\|] \{\gamma_q\|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\} \\ & + \overline{\mathcal{M}}_3[1 + \overline{\mathcal{M}}\mathcal{M}_5\|B\|^2 \|(G_{s_q}^{t_{q+1}})^{-1}\|] \{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} + \Lambda_2, \\ & t \in (s_q, t_{q+1}), q = 1, 2, \dots, p, \end{aligned}$$

where

$$\begin{aligned} \Lambda_2 = & \mathcal{M}_2(1 + \overline{\mathcal{M}}_1) + \overline{\mathcal{M}}\mathcal{M}_3 \|(G_{s_q}^{t_{q+1}})^{-1}\| \{\|w_{q+1}\| + \mathcal{M}_2(1 + \mathcal{M}_4)\}, \\ & t \in (s_q, t_{q+1}], q = 1, 2, \dots, p. \end{aligned}$$

$$\|\Upsilon_1(w, u)\| \leq \{\gamma_q\|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\}, \forall t \in (t_q, s_q], q = 1, 2, \dots, p.$$

Therefore, we sustain

$$\begin{aligned} & \|\widetilde{\Upsilon}(w, u)\| \\ & \leq \|\Upsilon_1(w, u)\| + \|\Upsilon_2(w, u)\| \\ & \leq [\widetilde{\mathcal{M}}_1 + \mathcal{M}_1\widetilde{\mathcal{M}}\|B\| \|(G_{s_0}^{t_1})^{-1}\| (1 + \widetilde{\mathcal{M}}_3\|B\|)] \{c\|w\|_{\mathcal{C}}^\eta + c_0\} + [\widetilde{\mathcal{M}}_3 \\ & \quad + \widetilde{\mathcal{M}}\mathcal{M}_3\|B\| (1 + \widetilde{\mathcal{M}}_3\|B\|) \|(G_{s_0}^{t_1})^{-1}\|] \{\gamma_0\|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} + \overline{\Lambda}_1, t \in [0, t_1], \\ & \|\widetilde{\Upsilon}(w, u)\| \end{aligned}$$

$$\begin{aligned}
 &\leq [\overline{\mathcal{M}}_1 + \|B\| \|(G_{s_q}^{t_{q+1}})^{-1} \|(\overline{\mathcal{M}} \mathcal{M}_5 + \mathcal{M} \mathcal{M}_4 \overline{\mathcal{M}}_3 \|B\|)] \{\gamma_q \|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\} \\
 &\quad + [\overline{\mathcal{M}}_3 + \overline{\mathcal{M}} \mathcal{M}_5 \|B\| \|(G_{s_q}^{t_{q+1}})^{-1} \|(1 + \overline{\mathcal{M}}_3 \|B\|)] \{\gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0} + \delta_0\} + \overline{\Lambda}_2, \\
 &\quad t \in (s_q, t_{q+1}], q = 1, 2, \dots, p, \\
 \|\tilde{\Upsilon}(w, u)\| &\leq \{\gamma_q \|w\|_{\mathbb{R}^n}^{\beta_q} + \delta_q\}, t \in (t_q, s_q], q = 1, 2, \dots, p,
 \end{aligned}$$

where

$$\begin{aligned}
 \overline{\Lambda}_1 &= \widetilde{\mathcal{M}}_1 \|B\| \|(G_{s_0}^{t_1})^{-1} \|(1 + \widetilde{\mathcal{M}}_3 \|B\|) \|w_{t_1}\| + \mathcal{M}_1 \widetilde{\mathcal{M}} \|B\| \|(G_{s_0}^{t_1})^{-1} \\
 (\times) &\|(1 + \widetilde{\mathcal{M}}_3 \|B\|) \|w_0\| + \widetilde{\mathcal{M}}_1 \|w_0\| + \mathcal{M}_2 (1 + \widetilde{\mathcal{M}}_1) + \widetilde{\mathcal{M}} \|B\| \|(G_{s_0}^{t_1})^{-1} \\
 &\quad (\times) \|(1 + \widetilde{\mathcal{M}}_3 \|B\|) [\mathcal{M}_2 (1 + \mathcal{M}_1)]
 \end{aligned}$$

and

$$\begin{aligned}
 \overline{\Lambda}_2 &= \overline{\mathcal{M}} \|B\| \|(G_{s_q}^{t_{q+1}})^{-1} \|\{\|w_{q+1}\| \\
 &\quad + \mathcal{M}_2 (1 + \mathcal{M}_4)\} [1 + \overline{\mathcal{M}}_3 \|B\|] + \mathcal{M}_2 (1 + \widetilde{\mathcal{M}}_1), q = 1, 2, \dots, p.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \frac{\|\tilde{\Upsilon}(w, u)\|}{\|(w, u)\|} &\leq \frac{\overline{\Lambda}_1}{\|w\| + \|u\|} + [\widetilde{\mathcal{M}}_1 + \mathcal{M}_1 \widetilde{\mathcal{M}} \|B\| \|(G_{s_0}^{t_1})^{-1} \|(1 + \widetilde{\mathcal{M}}_3 \|B\|)] \\
 &\quad (\times) \left\{ c \|w\|_{\mathcal{E}}^{\eta-1} + \frac{c_0}{\|w\| + \|u\|} \right\} + [\widetilde{\mathcal{M}}_3 + \widetilde{\mathcal{M}} \mathcal{M}_3 \|B\| (1 + \widetilde{\mathcal{M}}_3 \|B\|)] \\
 &\quad (\times) \|(G_{s_0}^{t_1})^{-1}\| \left\{ \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0-1} + \frac{\delta_0}{\|w\| + \|u\|} \right\}; \\
 \frac{\|\tilde{\Upsilon}(w, u)\|}{\|(w, u)\|} &\leq \frac{\overline{\Lambda}_2}{\|w\| + \|u\|} + [\overline{\mathcal{M}}_3 + \overline{\mathcal{M}} \mathcal{M}_5 \|B\| \|(G_{s_q}^{t_{q+1}})^{-1} \|(1 + \overline{\mathcal{M}}_3 \|B\|)] \\
 &\quad (\times) \left\{ \gamma_0 \|w\|_{\mathbb{R}^n}^{\beta_0-1} + \frac{\delta_0}{\|w\| + \|u\|} \right\} + [\overline{\mathcal{M}}_1 + \|B\| \|(G_{s_q}^{t_{q+1}})^{-1} \| \\
 &\quad (\times) (\overline{\mathcal{M}} \mathcal{M}_5 + \mathcal{M} \mathcal{M}_4 \overline{\mathcal{M}}_3 \|B\|)] \left\{ \gamma_q \|w\|_{\mathbb{R}^n}^{\beta_q-1} + \frac{\delta_q}{\|w\| + \|u\|} \right\}; \\
 \frac{\|\tilde{\Upsilon}(w, u)\|}{\|(w, u)\|} &\leq \left\{ \gamma_q \|w\|_{\mathbb{R}^n}^{\beta_q-1} + \frac{\delta_q}{\|w\| + \|u\|} \right\}.
 \end{aligned}$$

Hence, we have

$$\lim_{\|(w, u)\| \rightarrow \infty} \frac{\|\tilde{\Upsilon}(w, u)\|}{\|(w, u)\|} = 0.$$

**Step 4:** Finally, we will demonstrate that the operator  $\tilde{\Upsilon}$  has a fixed point.

Indeed, for a fixed  $0 < \alpha < 1$ , there exists  $r > 0$  large enough in a way that

$$\|\tilde{\Upsilon}(w, u)\| \leq \alpha\|(w, u)\| \quad \text{and} \quad \|(w, u)\| = r.$$

Let  $B(0, r)$  is a ball of radius  $r > 0$  and center zero, we have that  $\tilde{\Upsilon}(\partial B(0, r)) \subset B(0, r)$ . In view of the maps the sphere  $\partial B(0, r)$  into the interior of the ball  $B(0, r)$  and  $\tilde{\Upsilon}$  is compact.

Presently, we are able to utilize the Rothe's fixed point theorem to guarantee the existence of a fixed point  $(w, u) \in B(0, r) \subset \mathcal{PC}(\mathcal{J}, \mathbb{R}^n) \times C(\mathcal{J}, \mathbb{R}^m)$  such that

$$u = \bar{\Upsilon}\lambda_0(w, u) = B^*(\cdot)U^*(t_1, \cdot)(G_{s_0}^{t_1})^{-1}\lambda_0(w, u), \quad t \in [0, t_1].$$

Then

$$\begin{aligned} \Upsilon u &= \lambda_0(w, u) \\ &= w_{t_1} - U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] - \mathcal{H}(t_1, w(t_1)) \\ &\quad - \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_0^{t_1} U(t_1, s)B(s)u(s)ds \\ &= w_{t_1} - U(t_1, 0)[w_0 + r(w) - \mathcal{H}(0, w(0))] - \mathcal{H}(t_1, w(t_1)) \\ &\quad - \int_0^{t_1} U(t_1, s)\mathcal{F}(s, w(s))ds. \end{aligned}$$

Similarly for  $t \in (s_q, t_{q+1}]$ ,  $q = 1, 2, \dots, p$ , we have

$$\begin{aligned} u &= \bar{\Upsilon}\lambda_q(w, u) = B^*(\cdot)U^*(t_{q+1}, \cdot)(G_{s_q}^{t_{q+1}})^{-1}\lambda_q(w, u), \\ &\quad t \in (s_q, t_{q+1}], \quad q = 1, 2, \dots, p. \end{aligned}$$

Then

$$\begin{aligned} \Upsilon u &= \lambda_q(w, u) \\ &= w_{t_{q+1}} - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \\ &\quad - \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds. \end{aligned}$$



Therefore

$$\begin{aligned} & \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)B(s)u(s)ds \\ &= w_{t_{q+1}} - U(t_{q+1}, s_q)[\xi_q(s_q, w(t_q^-)) - \mathcal{H}(s_q, w(t_q^-))] - \mathcal{H}(t_{q+1}, w(t_q^-)) \\ & \quad - \int_{s_q}^{t_{q+1}} U(t_{q+1}, s)\mathcal{F}(s, w(s))ds. \end{aligned}$$

□

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